

EXPONENTIAL DICHOTOMY AND TRICHOTOMY FOR SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

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Abstract. The paper emphasizes the properties of exponential dichotomy and exponential trichotomy for skew-evolution semiflows in Banach spaces, by means of evolution semiflows and evolution cocycles. The approach is from uniform point of view. Some characterizations which generalize classic results are also provided.

Mathematics Subject Classification: 34D09

Keywords: Evolution semiflow, evolution cocycle, skew-evolution semiflow, exponential dichotomy, exponential trichotomy

1 Preliminaries

The exponential dichotomy is one of the basic concepts in the theory of dynamical systems and plays an important role in the study of stable and instable manifolds. Various concepts of dichotomy were introduced and studied by S.N. Chow and H. Leiva in [2], M. Megan, A.L. Sasu and B. Sasu in [4], P. Preda and C. Preda in [11]. A natural generalization of the notion of dichotomy is considered the trichotomy, which refers at a decomposition of the space at every moment into three closed subspaces: a stable subspace, an instable one and a center manifold. The trichotomy was introduced by R.J. Sacker and G.R. Sell in [12] and the exponential trichotomy by S. Elaydi and O. Hajek in [3]. In recent years interesting results in the domain of trichotomy were obtained by L.H. Popescu in [10], B. Sasu and A.L. Sasu in [13] or L. Barreira and C. Vallis in [1].

A new concept of trichotomy, the null uniform exponential trichotomy for evolution operators was introduced in [5]. The study has been continued by the definition of uniform exponential trichotomy by means of three projection families in [8]. The trichotomy is studied in the nonuniform setting for skew-evolution semiflows in [6] and [9] and for discrete time in [7].

In our paper we extend the asymptotic properties of exponential dichotomy and trichotomy for the newly introduced concept of skew-evolution semiflows, which can be considered generalization for evolution operators and skew-product semiflows.

2 Notations. Definitions. Examples

We consider (X, d) a metric space, V a Banach space and $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. We denote the sets $T = \{(t, t_0) \in \mathbb{R}^2, t \geq t_0 \geq 0\}$ and $Y = X \times V$. Let $P : Y \rightarrow Y$ be a projector given by $P(x, v) = (x, P(x)v)$, where $P(x)$ is a projection on $Y_x = \{x\} \times V$, $x \in X$.

Definition 2.1 A mapping $\varphi : T \times X \rightarrow X$ is called *evolution semiflow* on X if following relations hold:

- (s₁) $\varphi(t, t, x) = x, \forall (t, x) \in \mathbb{R}_+ \times X$
- (s₂) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall t \geq s \geq t_0 \geq 0, x \in X$.

Definition 2.2 A mapping $\Phi : T \times X \rightarrow \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if:

- (c₁) $\Phi(t, t, x) = I$, the identity operator on $V, \forall (t, x) \in \mathbb{R}_+ \times X$
- (c₂) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall t \geq s \geq t_0 \geq 0, x \in X$.

Definition 2.3 The mapping $C : T \times Y \rightarrow Y$ defined by the relation $C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$, where Φ is an evolution cocycle over an evolution semiflow φ , is called *skew-evolution semiflow* on Y .

Example 2.1 Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a decreasing function on $[0, \infty)$ such that there exists $\lim_{t \rightarrow \infty} f(t) = l$.

Let X be the closure in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of the set

$$\{f_t, t \in \mathbb{R}_+\}, \text{ where } f_t(\tau) = f(t + \tau), \forall \tau \in \mathbb{R}_+.$$

Then the mapping

$$\varphi : T \times X \rightarrow X, \varphi(t, s, x) = x_{t-s}$$

is an evolution semiflow on X .

Let us consider the Banach space $V = \mathbb{R}^p, p \geq 1$, with the norm $\|(v_1, \dots, v_p)\| = |v_1| + \dots + |v_p|$. The mapping

$$\Phi : T \times X \rightarrow \mathcal{B}(V), \Phi(t, s, x)(v_1, \dots, v_p) = \left(e^{\int_s^t x(\tau-s)d\tau} v_1, \dots, e^{\int_s^t x(\tau-s)d\tau} v_p \right)$$

is an evolution cocycle over φ and $C = (\varphi, \Phi)$ is a skew-evolution semiflow on Y .

Definition 2.4 Two projector families $\{P_k\}_{k \in \{1,2\}}$ are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

$$\begin{aligned} (dc_1) \quad & P_1(x) + P_2(x) = I, \quad P_1(x)P_2(x) = P_2(x)P_1(x) = 0 \\ (dc_2) \quad & P_k(\varphi(t, s, x))\Phi(t, s, x)v = \Phi(t, s, x)P_k(x)v, \quad k \in \{1, 2\} \end{aligned}$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Definition 2.5 A skew-evolution semiflow $C = (\varphi, \Phi)$ is *uniformly exponentially dichotomic* if there exist two projector families $\{P_k\}_{k \in \{1,2\}}$ compatible with C and some constants $N_1, N_2 \geq 1, \nu_1, \nu_2 > 0$ such that

$$\begin{aligned} (ued_1) \quad & e^{\nu_1(t-s)} \|\Phi(t, t_0, x)P_1(x)v\| \leq N_1 \|\Phi(s, t_0, x)P_1(x)v\| \\ (ued_2) \quad & e^{\nu_2(t-s)} \|\Phi(s, t_0, x)P_2(x)v\| \leq N_2 \|\Phi(t, t_0, x)P_2(x)v\| \end{aligned}$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Example 2.2 We consider X and the evolution semiflow φ as in Example 2.1. Let $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|$. The mapping

$$\Phi : T \times X \rightarrow \mathcal{B}(V), \quad \Phi(t, s, x)(v) = \left(v_1 e^{-2 \int_s^t x(\tau-s)d\tau}, v_2 e^{3 \int_s^t x(\tau-s)d\tau} \right)$$

is an evolution cocycle.

We consider the projectors $P_1(x, v) = (v_1, 0), P_2(x, v) = (0, v_2)$.

Then $C = (\varphi, \Phi)$ is a uniformly exponentially dichotomic skew-evolution semiflow with $N_1 = N_2 = 1, \nu_1 = 2, \nu_2 = 3$.

Definition 2.6 Three projector families $\{P_k\}_{k \in \{1,2,3\}}$ are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

$$(tc_1) \quad P_1(x) + P_2(x) + P_3(x) = I, \quad P_i(x)P_j(x) = P_j(x)P_i(x) = 0, \quad i, j \in \{1, 2, 3\}, \quad i \neq j$$

$$(tc_2) \quad P_k(\varphi(t, s, x))\Phi(t, s, x)v = \Phi(t, s, x)P_k(x)v, \quad k \in \{1, 2, 3\}$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Definition 2.7 A skew-evolution semiflow C is *uniformly exponentially tri-chotomic* if there exist three projector families $\{P_k\}_{k \in \{1,2,3\}}$ compatible with C and some constants $N_1, N_2, N_3 \geq 1, \nu_1, \nu_2, \nu_3 > 0$ such that

$$\begin{aligned} (uet_1) \quad & e^{\nu_1(t-s)} \|\Phi(t, t_0, x)P_1(x)v\| \leq N_1 \|\Phi(s, t_0, x)P_1(x)v\| \\ (uet_2) \quad & e^{\nu_2(t-s)} \|\Phi(s, t_0, x)P_2(x)v\| \leq N_2 \|\Phi(t, t_0, x)P_2(x)v\| \\ (uet_3) \quad & \|\Phi(t, t_0, x)P_3(x)v\| \leq N_3 e^{\nu_3(t-s)} \|\Phi(s, t_0, x)P_3(x)v\| \\ & \|\Phi(s, t_0, x)P_3(x)v\| \leq N_3 e^{\nu_3(t-s)} \|\Phi(t, t_0, x)P_3(x)v\| \end{aligned}$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Example 2.3 Let us consider function a , a metric space X and an evolution semiflow φ as in Example 2.1. Let $\mu > f(0)$.

We consider $V = \mathbb{R}^3$ with the norm $\|(v_1, v_2, v_3)\| = |v_1| + |v_2| + |v_3|$. The mapping

$$\Phi : T \times X \rightarrow \mathcal{B}(V), \quad \Phi(t, s, x)(v) =$$

$$= (e^{-\mu(t-t_0)+\int_{t_0}^t x(\tau-t_0)d\tau} v_1, e^{\int_{t_0}^t x(\tau-t_0)d\tau} v_2, e^{-(t-t_0)x(0)+\int_{t_0}^t x(\tau-t_0)d\tau} v_3)$$

is an evolution cocycle.

We consider the projectors $P_1(x, v) = (v_1, 0, 0)$, $P_2(x, v) = (0, v_2, 0)$, $P_3(x, v) = (0, 0, v_3)$.

Then $C = (\varphi, \Phi)$ is uniformly exponentially trichotomic with $N_1 = N_2 = N_3 = 1$, $\nu_1 = \mu - x(0)$, $\nu_2 = l$, $\nu_3 = x(0)$.

Remark 2.1 For $P_3 = 0$ we obtain in Definition 2.7 the property of uniform exponential dichotomy.

3 Main results

To characterize the uniform exponential dichotomy we will consider the next theorem.

Theorem 3.1 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is uniformly exponentially dichotomic if and only if there exist two projector families $\{P_k\}_{k \in \{1,2\}}$ compatible with C and a nondecreasing function $f : [0, \infty) \rightarrow (1, \infty)$ with the property $\lim_{t \rightarrow \infty} f(t) = \infty$ such that*

- (i) $f(t-s) \|\Phi(t, t_0, x)P_1(x)v\| \leq \|\Phi(s, t_0, x)P_1(x)v\|$
- (ii) $f(t-s) \|\Phi(s, t_0, x)P_2(x)v\| \leq \|\Phi(t, t_0, x)P_2(x)v\|$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Proof. Necessity. It is immediate if we consider $f(t) = N^{-1}e^{\nu t}$, $t \geq 0$, where $N = \max\{N_1, N_2\}$ and $\nu = \min\{\nu_1, \nu_2\}$, the constants N_1, N_2, ν_1, ν_2 being given by Definition 2.5.

Sufficiency. We will show that (i) implies (ued_1) . From the definition of function f there exists $\delta > 0$ such that $f(\delta) > 1$. We denote

$$\nu = \frac{\ln f(\delta)}{\delta} > 0.$$

Let $(t, s) \in T$. There exist $n \in \mathbb{N}$ and $r \in [0, \delta)$ such that $t - s = n\delta + r$. We have

$$\begin{aligned} e^{\nu(t-s)} \|\Phi(t, t_0, x)v\| &\leq f(\delta)[f(\delta)]^n \|\Phi(t, t_0, x)v\| \leq \\ &\leq f(\delta)[f(\delta)]^{n-1} \|\Phi(t - \delta, t_0, x)v\| \leq \dots \leq f(\delta) \|\Phi(t - n\delta, t_0, x)v\| \leq \\ &\leq f(\delta)f(r) \|\Phi(t - n\delta, t_0, x)v\| \leq f(\delta) \|\Phi(s, t_0, x)v\|. \end{aligned}$$

If we denote $N = f(\delta) > 1$, (ued_1) follows.

By an analogous deduction we obtain that (ii) implies (ued_2) .

Next result represent a characterization for the property of uniform exponential trichotomy.

Theorem 3.2 *Let $C = (\varphi, \Phi)$ be a skew-evolution semiflow with the property that for all $(t_0, x, v) \in \mathbb{R}_+ \times Y$ the mapping $s \mapsto \|\Phi(s, t_0, x)v\|$ is measurable. Then C is uniformly exponentially trichotomic if and only if there exist three projector families $\{P_k\}_{k \in \{1,2,3\}}$ compatible with C , some constants $N, M \geq 1$ and a nondecreasing function $g : [0, \infty) \rightarrow (1, \infty)$ with the property $\lim_{t \rightarrow \infty} g(t) = \infty$ such that*

- (i) $\|\Phi(t, t_0, x)P_1(x)v\| \leq N \|P_1(x)v\|$ and $\int_s^t \|\Phi(\tau, t_0, x)P_1(x)v\| d\tau \leq M \|\Phi(s, t_0, x)P_1(x)v\|$
- (ii) $\|P_2(x)v\| \leq N \|\Phi(t, t_0, x)P_2(x)v\|$ and $\int_s^t \|\Phi(\tau, t_0, x)P_2(x)v\| d\tau \leq M \|\Phi(t, t_0, x)P_2(x)v\|$
- (iii) $\|\Phi(t, t_0, x)P_3(x)v\| \leq g(t-s) \|\Phi(s, t_0, x)P_3(x)v\|$ and $\|\Phi(s, t_0, x)P_3(x)v\| \leq g(t-s) \|\Phi(t, t_0, x)P_3(x)v\|$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Proof. Necessity. It can be easily verified. We obtain $N = \max\{N_1, N_2\}$, $M = \max\{N_1\nu_1^{-1}, N_2\nu_2^{-1}\}$ and $g(t) = N_3^{-1}e^{\nu_3 t}$, the constants $N_1, N_2, N_3, \nu_1, \nu_2, \nu_3$ being given by Definition 2.7.

Sufficiency. We will prove that the relations in (i) imply (uet₁). We have

$$\begin{aligned} \|\Phi(t, t_0, x)P_1(x)v\| &= \|\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x)P_1(x)v\| \leq \\ &\leq N \|\Phi(s, t_0, x)P_1(x)v\|, \end{aligned} \quad (3.1)$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$. We integrate the obtained relation on $[s, t]$ and we have

$$\begin{aligned} (t-s) \|\Phi(t, t_0, x)P_1(x)v\| &\leq N \int_s^t \|\Phi(\tau, t_0, x)P_1(x)v\| d\tau \\ &\leq MN \|\Phi(s, t_0, x)P_1(x)v\|. \end{aligned} \quad (3.2)$$

We obtain, according to relations (3.1) and (3.2)

$$(t-s+1) \|\Phi(t, t_0, x)P_1(x)v\| \leq N(M+1) \|\Phi(s, t_0, x)P_1(x)v\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$. If we denote

$$f(u) = \frac{u+1}{N(M+1)}, \quad u \geq 0,$$

similarly as in the proof of Theorem 3.1, we obtain (uet₁).

An analogous deduction can be applied to prove that (ii) implies (uet₂).

To prove that the first relation in (iii) implies the first relation in (uet₃), we consider $t \geq s \geq t_0 \geq 0$. We denote $n = [t-s]$. We consider $N_3 = g(1) > 1$, $\nu_3 = \ln N_3 > 0$ and we obtain

$$\|\Phi(t, t_0, x)P_3(x)v\| \leq N_3 \|\Phi(t-1, t_0, x)P_3(x)v\| \leq \dots$$

$$\begin{aligned} \dots &\leq N_3^n \|\Phi(t-n, t_0, x)P_3(x)v\| \leq N_3^{n+1} \|\Phi(s, t_0, x)P_3(x)v\| = \\ &= N_3 e^{n\nu_3} \|\Phi(s, t_0, x)P_3(x)v\| \leq N_3 e^{\nu_3(t-s)} \|\Phi(s, t_0, x)P_3(x)v\| \end{aligned}$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Similarly is obtained the second relation in (uet_3) from the corresponding relation in (iii) .

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